HECKE TRIANGLE GROUPS AND SPECIAL HYPERBOLIC ELEMENTS

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Abstract. We study the action of the Hecke triangle groups G_q on $\lambda_q\mathbb{Q}(\lambda_q^2) \cup \{\infty\}$ with $\lambda_q = 2\cos(\pi/q)$. When $q = 18$, we show the existence of infinitely many distinct orbits of fixed points of special hyperbolic elements of G_q. We also find new orbits for several other values of q. These results provide new examples of special affine pseudo-Anosov homeomorphisms on the unfoldings of regular q-gons. In particular, on the unfolding of the regular 18-gon, there are infinitely many distinct Veech group orbits of directions invariant under a special affine pseudo-Anosov.

1. Introduction

The *Hecke triangle groups* G_q are an infinite family of lattices in $\mathrm{SL}(2, \mathbb{R})$ parametrized by an integer $q \geq 3$. The group G_q is generated by the two matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_q = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \tag{1}$$

where $\lambda_q = 2\cos(\pi/q)$. Each parabolic element of G_q fixes a unique *cusp* in $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$, and each hyperbolic element fixes a pair of distinct points in $\mathbb{P}^1(\mathbb{R})$. Basic longstanding open questions of interest about G_q include characterizing the set of cusps [Ros] and the set of hyperbolic fixed points as subsets of $\mathbb{P}^1(\mathbb{R})$.

The difficulty of characterizing the cusps of G_q seems to depend mainly on the degree of the *invariant trace field* of G_q, defined by

$$K_q = \mathbb{Q}(\mathrm{Tr}(A^2) : A \in G_q) = \begin{cases} \mathbb{Q}(\lambda_q), & q \text{ odd}, \\ \mathbb{Q}(\lambda_q^2), & q \text{ even}, \end{cases}$$

which is invariant under commensurability, see for instance [McM4]). The cusps of G_q form a single orbit $G_q \cdot \infty$. By an inductive argument using the generators in (1), we have

$$G_q \cdot \infty \subset \lambda_q\mathbb{Q}(\lambda_q^2) \cup \{\infty\}. \tag{2}$$

When q is odd, $\lambda_q\mathbb{Q}(\lambda_q^2) \cup \{\infty\}$ is equal to $\mathbb{P}^1(K_q)$. If $K_q = \mathbb{Q}$, that is, $q \in \{3, 4, 6\}$, then G_q is commensurable with $\mathrm{SL}(2, \mathbb{Z})$ and it is easy to show that the containment in (2) is an equality. When K_q is quadratic, equivalently $q \in \{5, 8, 10, 12\}$, there are several proofs showing that equality holds in these cases as well, using descent arguments or a study of the SAF-invariant [Leu], [McM1], [McM3], [Pan].

When K_q has degree at least 3, the question of characterizing the cusps of G_q is wide open, and it is known that the cusps are strictly contained in $\lambda_q\mathbb{Q}(\lambda_q^2) \cup \{\infty\}$, see [AS] and references therein. An important new phenomenon in this case is the existence of *special* hyperbolic elements of G_q, which have eigenvalues in K_q. In contrast, a typical hyperbolic element of G_q has eigenvalues in a quadratic extension of K_q. Fixed points of special hyperbolic elements of G_q lie in $\lambda_q\mathbb{Q}(\lambda_q^2)$ and thus provide new examples of G_q-orbits contained in $\lambda_q\mathbb{Q}(\lambda_q^2)$. Only a few examples of such orbits are known [AS], [HMTY], and in all known examples K_q has degree 3 or 4. One can also obtain lower bounds on the number of orbits of G_q in $\lambda_q\mathbb{Q}(\lambda_q^2)$ with algebraic methods, by studying the reduction of G_q modulo 2 or the class number of K_q [AS], [BR], [HMTY], [Wol].

However, in all cases where K_q has degree at least 3, it was previously unknown whether the number of orbits of G_q in $\lambda_q\mathbb{Q}(\lambda_q^2)$ was finite. Surprisingly, for at least one value of q, special hyperbolic elements are abundant enough to produce infinitely many orbits.

Theorem 1.1. For $q = 18$, there are infinitely many distinct G_q-orbits of fixed points of special hyperbolic elements of G_q contained in $\lambda_q\mathbb{Q}(\lambda_q^2)$.

Hyperbolic elements in G_q with a common power have the same fixed points, and conjugating a hyperbolic element in G_q only moves its fixed points within their respective G_q-orbits. Thus, Theorem 1.1 tells us there are infinitely many non-conjugate maximal cyclic subgroups of special hyperbolic elements in G_q when $q = 18$.

Let $\mathcal{O}_{K_q}$ be the ring of integers in K_q, and let $\mathcal{O}^*_{K_q}$ be its unit group. We searched for new examples of special hyperbolic elements of G_q by computing the λ_q-continued fraction expansions of many elements of $\lambda_q\mathcal{O}_{K_q}$ and $\lambda_q\mathcal{O}^*_{K_q}$. The examples we found (excluding $q = 18$) are shown in Tables 1 and 3 in terms of periodic λ_q-continued fractions. Table 3 contains some of the examples found for $q = 18$. In [HMTY], it was conjectured that there are exactly 2 orbits for G_7 in $\mathbb{P}^1(K_q)$, represented by ∞ and $\lambda_7^2 - 1$, and distinguished by residue classes modulo 2. A similar conjecture for the unit group $\mathcal{O}^*_{K_7}$ appears in [RT]. We expect that the structure of the G_q-orbits in $\lambda_q\mathbb{Q}(\lambda_q^2) \cup \{\infty\}$ is much more complicated, even for $q = 7$. For example, all but 27 (rational) integers in the interval $[1, 10^6]$ are cusps of G_7. The first few exceptional integers are

671, 26197, 98335, 121380, 221444, 249976, 255730, 298572, 327023, 327068, 339794, ...

and all 27 are special hyperbolic fixed points that do not lie in the orbit of $\lambda_7^2 - 1$. Regarding $\mathcal{O}^*_{K_7}$, letting $\lambda'_7 = -\lambda_7^2 + 2$, the unit $\lambda_7^7(-\lambda'_7)^{-23}$ is a special hyperbolic fixed point that does not lie in the orbit of $\lambda_7^2 - 1$.

Part of our motivation for studying the Hecke triangle groups G_q comes from the dynamics of billiards in the regular q-gon and straight-line flows on the translation surfaces (X_q, ω_q) obtained by unfolding these tables. We refer to [AS] for more details about the following discussion. The surface (X_q, ω_q) is obtained from one or two copies of a regular q-gon by gluing pairs of opposite parallel sides by translations. The derivatives of orientation-preserving affine automorphisms of (X_q, ω_q) form the *Veech group* V_q, which in this case is a lattice in $\mathrm{SL}(2, \mathbb{R})$ conjugate to G_q or an index 2 subgroup of G_q. As a consequence, these surfaces satisfy the *Veech dichotomy* [Vee]: every straight-line flow is either periodic or uniquely ergodic. However, the Veech dichotomy does not tell us which directions are periodic. The set of periodic directions for (X_q, ω_q) is precisely the set of cusps of V_q, which is in turn equal to the set of cusps of G_q up to acting by an element of $\mathrm{SL}(2, \mathbb{R})$.

The group $\mathrm{GL}^+(2, \mathbb{R})$ acts on the moduli space of all translation surfaces of a given genus. By applying an appropriate element $M \in \mathrm{GL}^+(2, \mathbb{R})$ to (X_q, ω_q), one can arrange that the periodic directions for $M(X_q, \omega_q)$ lie in $\mathbb{P}^1(K_q)$ [AS], [CS2]. With this normalization, the directions in $\mathbb{P}^1(K_q)$ are precisely the directions for which the straight-line flow has vanishing SAF-invariant. Roughly speaking, the SAF-invariant measures the algebraic obstruction to being periodic. Periodic straight-line flows have zero SAF-invariant, but it is well-known that the converse is not true. A striking source of uniquely ergodic flows with zero SAF-invariant are flows invariant under a *special* affine pseudo-Anosov homeomorphism. The first known examples are the Arnoux-Yoccoz pseudo-Anosovs [AY], and other families of examples are constructed in [CS1], [DS]. These pseudo-Anosovs are affine automorphisms whose derivative is a special hyperbolic element of the associated Veech group.

In the language of translation surfaces, Theorem 1.1 tells us that on the unfolding of the regular 18-gon, special affine pseudo-Anosovs are abundant. Moreover, this conclusion only depends on the commensurability class of the Veech group.

Theorem 1.2. For any translation surface (X, ω) whose Veech group V is commensurable to the G_{18} triangle group, there are infinitely many distinct V-orbits of directions that are invariant under a special affine pseudo-Anosov homeomorphism of (X, ω).

In particular, for a suitable $M \in \mathrm{GL}^+(2, \mathbb{R})$, the Veech group of $M(X_{18}, \omega_{18})$ acts on $\mathbb{P}^1(K_{18})$ and there are infinitely many distinct orbits in $\mathbb{P}^1(K_{18})$ for this action. To the best of our knowledge, Theorem 1.2 provides the first known examples of lattice Veech groups that act on the projective line over their invariant trace field with infinitely many orbits. For a lattice Veech group, the invariant trace field is equal to its *trace field*, the field generated over $\mathbb{Q}$ by the traces [Hoo]. Note that by [McM1], for any lattice Veech group with a rational or quadratic trace field K, after normalizing every element of $\mathbb{P}^1(K)$ is a cusp and there are only finitely many orbits of cusps. We also remark that the special pseudo-Anosovs in Theorem 1.2 lie in infinitely many distinct conjugacy classes in the mapping class group Mod_g, where g is the genus of X.

Our proof of Theorem 1.1 is obtained via explicit constructions. For example, letting $T = T_{18}$, we will see that

$$\left(ST^4ST^{-1}ST^{-4}ST\right)^k ST^2ST^{-2}ST^{-2}\left(STST^4ST^{-1}ST^{-4}\right)^k ST^2 \tag{3}$$

is a special hyperbolic element of G_{18} for all $k \geq 0$. In the next section, we will provide several other infinite families of special hyperbolic elements. For any q, the eigenvalues of a special hyperbolic element of G_q lie in the unit group $\mathcal{O}^*_{K_q}$. When $q = 18$, we have $\mathcal{O}^*_{K_{18}} \cong \mathbb{Z}/2 \times \mathbb{Z}^2$. Despite this, all of the special hyperbolic elements we found in this case had eigenvalues lying in a single cyclic group up to sign. In particular, it turns out that the eigenvalues of the special hyperbolic elements in (3) are all powers of the unit

$$u_{18} = 2\lambda_{18}^4 - 4\lambda_{18}^2 + 1. \tag{4}$$

We observed similar coincidences for other values of q, especially $q = 7$. In all but one case, the eigenvalues of the special hyperbolic elements in G_7 we found were powers of the unit

$$u_7 = \lambda_7^2 + \lambda_7 \tag{5}$$

up to sign, which leads us to suspect that similar constructions to (3) are possible for some other values of q. A partial analogue of this phenomenon is known to arise in real quadratic fields via the classical continued fractions associated to $G_3 = \mathrm{SL}(2, \mathbb{Z})$, see [McM2] and [Wil] for constructions. However, we emphasize that the associated hyperbolic elements of G_3 are not special, indeed G_3 does not contain any special hyperbolic elements.

More generally, Theorem 1.1, the occurrence of "rare" G_q-orbits as shown in Tables 1 and 2, and the behavior shown in Figure 1 (right), seem to suggest a positive answer to the following question.

Question 1.3. For all q such that K_q has degree at least 3, are there infinitely many distinct G_q-orbits contained in $\lambda_q\mathbb{Q}(\lambda_q^2)$?

Lastly, to emphasize how poorly understood the sets of cusps and hyperbolic fixed points of G_q still are, the following questions are open in all cases where K_q has degree at least 3.

(1) Is there an element of $\lambda_q\mathbb{Q}(\lambda_q^2)$ that is not fixed by any element of G_q?
(2) Is there an (always terminating) algorithm that takes as input $x \in \lambda_q\mathbb{Q}(\lambda_q^2)$ and outputs whether x is a cusp of G_q? A hyperbolic fixed point of G_q?
(3) For q odd, does G_q have infinitely many integer cups?

In [Bou] and [HMTY], it is conjectured that question (1) has a negative answer for $q = 7, 9$. Based on our searches, we expect that question (1) also has a negative answer for $q = 18$, and possibly for $q = 14, 30$ as well. Note that for each q, a negative answer to (1) implies a positive answer to (2). For question (3), our searches suggest a positive answer for $q = 7, 9$.

Acknowledgements. The author thanks Curt McMullen for many inspiring discussions on this topic. This work was primarily carried out while the author was supported by an NSF GRFP under grant DGE-1144152. The author also acknowledges support from the NSF under grant DMS-2303185.

Special hyperbolic elements of G_{18}

We recall material about Hecke triangle groups, their action on $\mathbb{P}^1(\mathbb{R})$ via Möbius transformations, and traces of matrix products in $\mathrm{SL}(2,\mathbb{R})$. We then prove Theorem 1.1.

Stabilizers and conjugacy classes in G_q. A matrix $A \in \mathrm{SL}(2,\mathbb{R})$ is *hyperbolic* if $|\operatorname{Tr}(A)| > 2$, *parabolic* if $|\operatorname{Tr}(A)| = 2$, and *elliptic* if $|\operatorname{Tr}(A)| < 2$. Similarly for elements of $\mathrm{PSL}(2,\mathbb{R})$. As a subgroup of $\mathrm{SL}(2,\mathbb{R})$, the Hecke triangle group G_q acts on $\mathbb{P}^1(\mathbb{R})$ by Möbius transformations with $\pm I$ acting trivially. We denote by $\overline{G}_q$ the image of G_q in $\mathrm{PSL}(2,\mathbb{R})$.

The stabilizer in $\overline{G}_q$ of a point in $\mathbb{P}^1(\mathbb{R})$ is cyclic (Theorem 8.1.2 in [Bea]). The stabilizer of a hyperbolic fixed point is a maximal cyclic subgroup consisting of hyperbolic elements. *Primitive* hyperbolic elements are the generators of these stabilizers. By sending a hyperbolic element to its attracting fixed point, we obtain a bijection between primitive hyperbolic elements of $\overline{G}_q$ and hyperbolic fixed points, and similarly a bijection between conjugacy classes of primitive hyperbolic elements in $\overline{G}_q$ and orbits of hyperbolic fixed points. We record this discussion with the following lemma.

Lemma 1.4. If $A_1, A_2 \in G_q$ are hyperbolic elements whose attracting fixed points lie in the same G_q-orbit, then there is $B \in G_q$ and integers $m_1, m_2 > 0$ such that $BA_1^{m_1}B^{-1} = \pm A_2^{m_2}$.

The group $\overline{G}_q$ is isomorphic to a free product of two cyclic groups of orders 2 and q, respectively. The images in $\mathrm{PSL}(2,\mathbb{R})$ of the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U_q = ST_q = \begin{pmatrix} 0 & -1 \\ 1 & \lambda_q \end{pmatrix}$$

realize the presentation

$$\overline{G}_q = \langle S, U_q \mid S^2 = U_q^q = I \rangle.$$

Denote $T = T_q$ and $U = U_q$. Each element in $\overline{G}_q$ can thus be expressed uniquely as a *reduced* word in S and U (Theorem 4.1 in [MKS]). These reduced words are powers of S or U or have the form

$$S^{\varepsilon_1} U^{a_1} S U^{a_2} \cdots S U^{a_{k-1}} S U^{a_k} S^{\varepsilon_k}$$

for some $k > 0$, $1 \le a_1, \ldots, a_k \le q-1$, and $\varepsilon_1, \varepsilon_k \in \{0,1\}$. A reduced word is *cyclically reduced* if it is a power of S or U or if $\varepsilon_1 \ne \varepsilon_k$ above. Up to conjugation by S, a cyclically reduced word is a power of S or U or has the form

$$SU^{a_1} \cdots SU^{a_k}$$

for some $k > 0$ and $1 \le a_1, \ldots, a_k \le q-1$. Two cyclically reduced words of this form are conjugate in $\overline{G}_q$ if and only if they are cyclic permutations of each other (Theorem 4.2 in [MKS]). For convenience, we will use a weaker version of this fact for words in S, T.

Lemma 1.5. Consider elements of $\overline{G}_q$ of the form

$$ST^{n_1} \cdots ST^{n_k} \tag{6}$$

with $k > 0$, $n_j \ne 0$ for $1 \le j \le k$ and not all the same sign, and such that in the cyclically ordered sequence $n_1, \ldots, n_k$, the maximum number of consecutive 1's or consecutive (-1)'s is less than $q/2 - 2$. Two such elements are conjugate in $\overline{G}_q$ if and only if they are cyclic permutations of each other.

Proof. For each $1 \le j \le k$ such that $n_j \ge 2$ or $n_j \le -2$, substitute ST^{n_j} with $U(SU)^{n_j - 1}$ or $S(U^{q-1}S)^{-n_j}$, respectively. For each maximal sequence of consecutive 1's in the cyclically ordered sequence $n_1, \ldots, n_k$ of length r, substitute $(ST)^r$ with U^r. For each maximal sequence of consecutive (-1)'s of length r, substitute $(ST^{-1})^r$ with $SU^{q-r}S$. We have rewritten $ST^{n_1} \cdots ST^{n_k}$ as a concatenation of words $W_1 \cdots W_\ell$ with $1 \le \ell \le k$. Each word W_s is a reduced word in S, U, and the words coming from positive integers in $n_1, \ldots, n_k$ begin and end with U, while the words coming from negative integers begin and end in S.

If $s, s+1, \ldots, s+t$ is a maximal sequence such that $W_s, W_{s+1}, \ldots, W_{s+t}$ all come from positive integers, then the concatenation $W_s W_{s+1} \cdots W_{s+t}$ has the form

$$U^{b_0} U(SU)^{a_1 - 1} U^{b_1} U(SU)^{a_2 - 1} U^{b_2} \cdots U(SU)^{a_m - 1} U^{b_m}$$

with $a_1, \ldots, a_m \ge 2$ and $0 \le b_0, \ldots, b_m < q/2 - 2$. Combining adjacent U's gives us

$$U^{b_0+1}(SU)^{a_1-2} SU^{b_1+2}(SU)^{a_2-2} SU^{b_2+2} \cdots (SU)^{a_m-2} SU^{b_m+1} \tag{7}$$

which is reduced after removing $(SU)^{a_i-2}$ when $a_i - 2 = 0$. If $s, s+1, \ldots, s+t$ is a maximal sequence such that $W_s, W_{s+1}, \ldots, W_{s+t}$ all come from negative integers, then the concatenation $W_s W_{s+1} \cdots W_{s+t}$ has the form

$$(SU^{q-b_0}S)S(U^{q-1}S)^{a_1}(SU^{q-b_1}S)S(U^{q-1}S)^{a_2}(SU^{q-b_2}S) \cdots S(U^{q-1}S)^{a_m}(SU^{q-b_m}S)$$

with $a_1, \ldots, a_k \ge 2$, $0 \le b_0, \ldots, b_m < q/2 - 2$. Cancelling adjacent S's and then combining adjacent U's gives us

$$SU^{q-b_0-1}S(U^{q-1}S)^{a_1-2}U^{q-b_1-2}S(U^{q-1}S)^{a_2-2}U^{q-b_2-2}S \cdots (U^{q-1}S)^{a_m-2}SU^{q-b_m-1}S \tag{8}$$

which is reduced after removing $(U^{q-1}S)^{a_j-2}$ when $a_j - 2 = 0$. Since the reduced words in (7) begin and end in U, and the reduced words in (8) begin and end in S, a concatenation of such words that alternates between (8) and (7) is also reduced. In this way, we can rewrite $ST^{n_1} \cdots ST^{n_k}$ as a reduced word in S, U.

Suppose two elements $ST^{n_1} \cdots ST^{n_k}$ and $ST^{m_1} \cdots ST^{m_\ell}$ of G_q as in (6) are conjugate in G_q. Since $n_1, \ldots, n_k$ do not all have the same sign, by conjugating by $ST^{n_1} \cdots ST^{n_j}$ if necessary we may assume $n_1 < 0$ and $n_k > 0$, and we may similarly assume $m_1 < 0$ and $m_\ell > 0$. The associated reduced words in S, U are also conjugate, and by our sign assumptions, they are both cyclically reduced words beginning with S and ending in U, and thus are cyclic permutations of each other. Now, the powers U^p that appear in (7) all satisfy $1 \leq p < q/2$, and they determine the associated maximal sequence of positive integers in $n_1, \ldots, n_k$, with b_j the lengths of maximal sequences of consecutive 1's and a_j the integers greater than 1. Similarly, the powers U^p that appear in (8) all satisfy $q/2 < p \leq q-1$, and they determine the associated maximal sequence of negative integers in $n_1, \ldots, n_k$ with b_j the lengths of maximal sequences of consecutive (-1)'s and a_j the integers less than -1. Thus, since these cyclically reduced words in S, U are cyclic permutations of each other, the associated sequences $n_1, \ldots, n_k$ and $m_1, \ldots, m_\ell$ must be cyclic permutations of each other. □

Trace relations. Recall that for any matrix $A \in \mathrm{SL}(2, \mathbb{R})$,

$$A + A^{-1} = \mathrm{Tr}(A) I.$$

It follows that for any $A, B \in \mathrm{SL}(2, \mathbb{R})$,

$$\mathrm{Tr}(AB) + \mathrm{Tr}(AB^{-1}) = \mathrm{Tr}(A)\,\mathrm{Tr}(B). \tag{9}$$

We will be interested in the traces of certain matrix products of the form

$$M_{k,\ell} = D^\ell C B^k A$$

with $A, B, C, D \in \mathrm{SL}(2, \mathbb{R})$ and $k, \ell \in \mathbb{Z}$.

Lemma 1.6. Suppose $\mathrm{Tr}(B) = \mathrm{Tr}(D)$. Then for all integers $k \geq 3$,

$$\mathrm{Tr}(M_{k,k}) = (\mathrm{Tr}(B)^2 - 1)(\mathrm{Tr}(M_{k-1,k-1}) - \mathrm{Tr}(M_{k-2,k-2})) + \mathrm{Tr}(M_{k-3,k-3}).$$

Proof. Applying (9) and cyclic commutivity of trace, we get

$$\mathrm{Tr}(M_{k,\ell}) = \mathrm{Tr}(B)\,\mathrm{Tr}(M_{k-1,\ell}) - \mathrm{Tr}(M_{k-2,\ell})$$

for all $k \geq 2$, $\ell \geq 0$, and similarly

$$\mathrm{Tr}(M_{k,\ell}) = \mathrm{Tr}(B)\,\mathrm{Tr}(M_{k,\ell-1}) - \mathrm{Tr}(M_{k,\ell-2})$$

for all $k \geq 0$, $\ell \geq 2$. It follows that for all $k \geq 2$,

$$\begin{aligned} \mathrm{Tr}(M_{k,k}) = {} & \mathrm{Tr}(B)^2\,\mathrm{Tr}(M_{k-1,k-1}) + \mathrm{Tr}(M_{k-2,k-2}) \\ & - \mathrm{Tr}(B)(\mathrm{Tr}(M_{k-1,k-2}) + \mathrm{Tr}(M_{k-2,k-1})) \end{aligned} \tag{10}$$

and that

$$\mathrm{Tr}(M_{k,k-1}) + \mathrm{Tr}(M_{k-1,k}) = 2\,\mathrm{Tr}(B)\,\mathrm{Tr}(M_{k-1,k-1}) - (\mathrm{Tr}(M_{k-1,k-2}) + \mathrm{Tr}(M_{k-2,k-1})). \tag{11}$$

Now suppose $k \geq 3$. By applying (10) to $\mathrm{Tr}(M_{k,k})$ and $\mathrm{Tr}(M_{k-1,k-1})$, and applying (11) to $\mathrm{Tr}(M_{k-1,k-2}) + \mathrm{Tr}(M_{k-2,k-1})$, we get

$$\begin{aligned} \mathrm{Tr}(M_{k,k}) &= \mathrm{Tr}(B)^2\,\mathrm{Tr}(M_{k-1,k-1}) - \mathrm{Tr}(B)(\mathrm{Tr}(M_{k-1,k-2}) + \mathrm{Tr}(M_{k-2,k-1})) + \mathrm{Tr}(M_{k-2,k-2}) \\ &= \mathrm{Tr}(B)^2\,\mathrm{Tr}(M_{k-1,k-1}) + (1 - 2\,\mathrm{Tr}(B)^2)\,\mathrm{Tr}(M_{k-2,k-2}) \\ &\quad + \mathrm{Tr}(B)(\mathrm{Tr}(M_{k-2,k-3}) + \mathrm{Tr}(M_{k-3,k-2})) \\ &= (\mathrm{Tr}(B)^2 - 1)\,\mathrm{Tr}(M_{k-1,k-1}) + (1 - \mathrm{Tr}(B)^2)\,\mathrm{Tr}(M_{k-2,k-2}) \\ &\quad + \mathrm{Tr}(M_{k-1,k-1}) - \mathrm{Tr}(B)^2\,\mathrm{Tr}(M_{k-2,k-2}) + \mathrm{Tr}(B)(\mathrm{Tr}(M_{k-2,k-3}) + \mathrm{Tr}(M_{k-3,k-2})) \\ &= (\mathrm{Tr}(B)^2 - 1)(\mathrm{Tr}(M_{k-1,k-1}) - \mathrm{Tr}(M_{k-2,k-2})) + \mathrm{Tr}(M_{k-3,k-3}). \end{aligned}$$

□

Lemma 1.7. Suppose that for some integer $n \geq 0$,

$$\mathrm{Tr}(M_{0,0}) = \mathrm{Tr}(B^n), \quad \mathrm{Tr}(M_{1,1}) = \mathrm{Tr}(B^{n+2}), \quad \mathrm{Tr}(M_{2,2}) = \mathrm{Tr}(B^{n+4}).$$

Then for all integers $k \geq 3$,

$$\mathrm{Tr}(M_{k,k}) = \mathrm{Tr}(B^{n+2k}).$$

Proof. Let t, t^{-1} be the eigenvalues of B, so that $\mathrm{Tr}(B^m) = t^m + t^{-m}$ for all $m \in \mathbb{Z}$. We induct on k, and we may assume $k \geq 3$. By Lemma 1.6 and induction on k,

$$\begin{aligned}\mathrm{Tr}(M_{k,k}) = (t^2 + 1 + t^{-2})&(t^{n+2(k-1)} + t^{-n-2(k-1)} - t^{n+2(k-2)} - t^{-n-2(k-2)}) \\ &+ (t^{n+2(k-3)} + t^{-n-2(k-3)}).\end{aligned}$$

Expanding the right-hand side and canceling terms reduces to

$$\mathrm{Tr}(M_{k,k}) = t^{n+2k} + t^{-n-2k} = \mathrm{Tr}(C^{n+2k}).$$

□

Families of special hyperbolic elements. A hyperbolic element of G_q is *special* if its eigenvalues lie in K_q. Denote $\lambda = \lambda_q$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_q$ be a special hyperbolic element, and let $t, t^{-1} \in K_q$ be its eigenvalues. Note that $c \neq 0$ since A does not fix the cusp ∞. Since A is special, $\mathrm{Tr}(A)^2 - 4 = (t - t^{-1})^2$ is a square in K_q. Solving $A \cdot x = (ax+b)/(cx+d) = x$, we see that the fixed points of A are given by $x = ((a-d) \pm (t - t^{-1}))/2c$. For q odd, since the matrix entries of G_q lie in $\mathbb{Z}[\lambda]$, we have $x \in \mathbb{Q}(\lambda) = \lambda\mathbb{Q}(\lambda^2)$. For q even, by an inductive argument using the generators S, T_q, the matrix A has one of the forms

$$\begin{pmatrix} a & b_0\lambda \\ c_0\lambda & d \end{pmatrix}, \quad \begin{pmatrix} a_0\lambda & b \\ c & d_0\lambda \end{pmatrix},$$

with $a_0, b_0, c_0, d_0 \in \mathbb{Z}[\lambda^2]$ (see Corollary 1 in [Ros]). Since $\mathrm{Tr}(A) \in K_q = \mathbb{Q}(\lambda^2)$ is nonzero, A is of the first form and $x = ((a-d) \pm (t - t^{-1}))/2\lambda c_0 \in \lambda\mathbb{Q}(\lambda^2)$. Thus, fixed points of special hyperbolic elements of G_q lie in $\lambda\mathbb{Q}(\lambda^2)$.

We now apply Lemma 1.7 to produce infinite families of orbits of special hyperbolic fixed points. Denote $T = T_{18}$ and $\lambda = \lambda_{18}$. For $n_1, \ldots, n_k$ a finite sequence of nonzero integers, define

$$M(n_1, \ldots, n_k) = ST^{n_k} \cdots ST^{n_1} \in G_{18}.$$

Additionally, denote by $(n_1, \ldots, n_k)^m$ the concatenation of m copies of $n_1, \ldots, n_k$. Lastly, recall the unit $u_{18} = 2\lambda^4 - 4\lambda^2 + 1$ from (4) and denote $u = u_{18}$. All of the computational checks in the proof of the following theorem were carried out with SageMath.

Theorem 1.8. For all integers $k \geq 0$, the following elements of G_{18} are special hyperbolic elements. The fixed points of each family form infinitely many distinct G_{18}-orbits in $\lambda\mathbb{Q}(\lambda^2)$.

$$M(2, (-4, -1, 4, 1)^k, -2, -2, 2, (1, -4, -1, 4)^k)$$
$$M(4, (2, -2, -2, 2)^k, 1, -4, -1, (2, 2, -2, -2)^k)$$
$$M(-4, (-1, 8, 1, -2)^k, -2, 1, 2, (-2, -1, 8, 1)^k)$$
$$M(-1, (-4, 2, 1, -2)^k, -2, 1, 8, (1, -1, -1, 16)^k)$$
$$M(16, (1, -2, -1, 8)^k, -1, -1, 1, (8, 1, -2, -1)^k)$$
$$M(4, (2, -2, -2, 2)^k, 2, -2, -1, 4, 1, -2, -2, (2, 2, -2, -2)^k)$$
$$M(4, (1, -2, -4, 2)^k, 1, -2, -2, 4, 2, -2, -1, (2, 4, -2, -1)^k)$$
$$M(2, (-4, -1, 4, 1)^k, -4, -1, 2, 2, -2, -1, 4, (1, -4, -1, 4)^k)$$
$$M(2, (-2, -1, 8, 1)^k, -2, -1, 4, 2, -4, -1, 2, (1, -8, -1, 2)^k)$$
$$M(-4, (-1, 8, 1, -2)^k, -1, 8, -1, -1, 1, 8, 1, (-2, -1, 8, 1)^k)$$
$$M(2, (-8, -1, 2, 1)^k, -8, -1, 1, 2, -1, -1, 8, (1, -2, -1, 8)^k)$$
$$M(2, (-1, -1, 16, 1)^k, -1, -1, 8, 2, -8, -1, 1, (1, -16, -1, 1)^k)$$
$$M(16, (1, -2, -1, 8)^k, 1, -2, -2, 1, 2, -2, -1, (8, 1, -2, -1)^k)$$

Proof. The proofs for each family are similar, so we only present the proof for the first family

$$M(2, (-4, -1, 4, 1)^k, -2, -2, 2, (1, -4, -1, 4)^k)$$

in detail. Up to cyclic permutation, the repeating parts in this family are equal to

$$M(4, 1, -4, -1) = \begin{pmatrix} 4\lambda^2 + 1 & 16\lambda^3 \\ 4\lambda^3 & 16\lambda^4 - 4\lambda^2 + 1 \end{pmatrix}$$

and the non-repeating part (the $k = 0$ case) is equal to

$$M(2, 2, -2, -2) = \begin{pmatrix} 4\lambda^2 + 1 & 8\lambda^3 \\ 8\lambda^3 & 16\lambda^4 - 4\lambda^2 + 1 \end{pmatrix}$$

and both of these matrices have trace

$$16\lambda^4 + 2 = u^2 + u^{-2}.$$

This family has the form

$$M_k = D^k C B^k A, \quad k \geq 0,$$

with $A = M(2)$, $B = M(4, 1, -4, -1)$, $C = M(-2, -2, 2)$, $D = M(1, -4, -1, 4)$. By the above calculations,

$$\mathrm{Tr}(M_0) = \mathrm{Tr}(B) = \mathrm{Tr}(D) = u^2 + u^{-2}.$$

Additionally, we check using Sage that

$$\mathrm{Tr}(M_1) = 1847952\lambda^4 - 3838464\lambda^2 + 1391618 = u^6 + u^{-6}$$

$$\mathrm{Tr}(M_2) = 110983509904\lambda^4 - 235185378816\lambda^2 + 85747037186 = u^{10} + u^{-10}$$

and thus by Lemma 1.7,

$$\mathrm{Tr}(M_k) = u^{4k+2} + u^{-4k-2}$$

for all $k \geq 0$. Since $u \in K_q$ and $|u| \approx 15.582 > 1$, this means M_k is a special hyperbolic element of G_{18} for all $k \geq 0$.

Suppose that $k_1, k_2 \geq 0$ are integers such that the attracting fixed points of M_{k_1}, M_{k_2} lie in the same G_{18}-orbit. By Lemma 1.4, there are integers $m_1, m_2 > 0$ such that $M_{k_1}^{m_1}$ and $\pm M_{k_2}^{m_2}$ are conjugate in G_{18}. The sequences

$$(2, (-4, -1, 4, 1)^{k_j}, -2, -2, 2, (1, -4, -1, 4)^{k_j})^{m_j}$$

defining $M_{k_j}^{m_j}$ do not contain multiple consecutive 1's or (-1)'s, so Lemma 1.5 tells us that these sequences are cyclic permutations of each other. By counting the number of 2's in each sequence, we see that $2m_1 = 2m_2$ and thus $m_1 = m_2$. Then by counting the number of 4's, we see that $2k_1m_1 = 2k_2m_2 = 2k_2m_1$ and thus $k_1 = k_2$. Thus, the attracting fixed points of M_k, $k \geq 0$, lie in infinitely many distinct G_{18}-orbits. □

λ_q-continued fraction expansions

Many questions about the Hecke triangle groups G_q can be studied computationally using a family of continued fraction algorithms introduced by Rosen [Ros]. Throughout, we denote $\lambda = \lambda_q$, $K = K_q$, and $T = T_q$. Rosen showed that any $x \in \mathbb{R}$ can be expressed as a λ-*continued fraction*

$$x = [a_0, a_1, a_2, \dots] = a_0\lambda - \cfrac{1}{a_1\lambda - \frac{1}{a_2\lambda - \cdots}} \tag{12}$$

with $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{Z} \setminus \{0\}$ for $i \geq 1$, in a unique way by requiring

$$\begin{aligned} x - a_0\lambda &\in (-\lambda/2, \lambda/2] \\ -1/(x - a_0\lambda) - a_1\lambda &\in (-\lambda/2, \lambda/2] \\ &\vdots \end{aligned}$$

The cusps $G_q \cdot \infty$ are precisely the elements of $\mathbb{P}^1(\mathbb{R})$ that can be expressed as finite λ-continued fractions $[a_0, a_1, \dots, a_N]$. If there exists $n \geq 1$ and $i_0 \geq 0$ such that $a_i = a_{i+n}$ for all $i \geq i_0$, we say that the λ-continued fraction is *preperiodic*, and *periodic* if we can take $i_0 = 0$. If x is a periodic λ-continued fraction, we denote this by

$$x = \overline{[a_0, a_1, \dots, a_n]}$$

where n is the minimal period, and x is fixed by

$$M_x = ST^{-a_n} \cdots ST^{-a_0} \in G_q.$$

For all $x \in \mathbb{P}^1(\mathbb{R})$, we have

$$S \cdot (-x) = -(S \cdot x), \quad T \cdot (-x) = -(T^{-1} \cdot x).$$

It follows that the set of cusps and the set of hyperbolic fixed points are preserved under negation. Since $\lambda\mathbb{Q}(\lambda^2)$ is invariant under negation, the set of special hyperbolic fixed points is also preserved under negation. Since $\lambda/2$ is a cusp [Ros], for any periodic λ-continued fraction, negation simply negates its periodic part. Suppose $x = \overline{[a_0, a_1, \dots, a_n]}$ is fixed by a hyperbolic element $M_x = ST^{-a_n} \cdots ST^{-a_0}$, and let $x' = \overline{[a_n, a_{n-1}, \dots, a_0]}$. Since M_x and M_x^{-1} have the same fixed points and $SM_x^{-1}S^{-1} = ST^{a_0} \cdots ST^{a_n}$, we see that M_x also fixes $S \cdot (-x') = 1/x'$. The periodic λ-continued fractions obtained by negating and reversing the periodic part do not necessarily lie in the same G_q-orbit. In this way, we may obtain 1, 2, or 4 distinct G_q-orbits.

Below, we summarize the computer searches we carried out to find new G_q-orbits of special hyperbolic fixed points. All of our computations were done using SageMath, which supports arithmetic in number fields and arbitrary precision numerical computations.

G_7-orbits. Let $q = 7$. Then λ has one other Galois conjugate with absolute value greater than 1, given by $\lambda' = -\lambda^2 + 2$. The ring of integers of K_q is $\mathbb{Z}[\lambda]$, and the positive unit group is generated by $\lambda, -\lambda'$. We counted the number of elements of each G_7-orbit in $\mathbb{Z} \cap [1, 10^6]$ by computing the associated λ-continued fraction expansions, and similarly for a large subset of the positive unit group and the ring of integers. The results are reported in Table 1. All of the λ-continued fraction expansions we computed were either finite or preperiodic.

We found several new G_7-orbits of special hyperbolic fixed points, and Table 1 lists the associated periodic λ-continued fractions, up to negation and reversal. We obtain a lower bound on the number N_7 of G_7-orbits in $\mathbb{P}^1(K_q)$ of

$$N_7 \geq 1 + 1 + 4 + 4 + 4 + 4 + 2 + 4 = 24$$

Note that only 13 of these 24 orbits appear in the Table 1. In particular, even up to negation and reversal of periodic parts, the orbit of

$$\frac{531}{7}\lambda^2 + \frac{402}{7}\lambda - \frac{319}{7} = \overline{[169, 1, 2, -1, -2, -1]}$$

does not appear. We found this orbit of special hyperbolic fixed points by checking all periodic λ-continued fractions of the form $\overline{[a_1, a_2, \ldots, a_6]}$ with $\prod_{j=1}^{6} a_i = \pm 2^2 \cdot 13^2$. Lastly, for all but the last row in the $q = 7$ section of Table 1, the eigenvalues of the associated special hyperbolic elements are powers of the unit

$$u_7 = \lambda_7^2 + \lambda_7.$$

We were unable to prove a version of Theorem 1.8 in the $q = 7$ case though.

G_9-orbits. Let $q = 9$. Again, the ring of integers of K_q is $\mathbb{Z}[\lambda]$, and the positive unit group is generated by λ and $-\lambda' = \lambda^2 - 2$. Table 2 counts the number of elements of each G_9-orbit in a large subset of the rational integers, the positive units, and the ring of integers, and Table 1 lists the associated periodic λ-continued fractions. We get a lower bound on the number N_9 of G_9-orbits in $\mathbb{P}^1(K_9)$ of

$$N_9 \geq 1 + 4 + 4 + 4 = 13.$$

All of the λ-continued fraction expansions we computed were either finite or preperiodic.

G_{18}-orbits. Let $q = 18$. We carried out similar searches as in the G_7 and G_9 cases, for elements of the form

$$\begin{aligned} \lambda n, &\quad n \in \mathbb{Z} \cap [1, 10^6] \\ \lambda\lambda_1^a\lambda_2^b, &\quad a, b \in \mathbb{Z} \cap [-10^2, 10^2] \\ \lambda^3 a + \lambda^5 b, &\quad a, b \in \mathbb{Z} \cap [-10^3, 10^3] \end{aligned}$$

where $\lambda_1 = \lambda^2 - 1$ and $\lambda_2 = \lambda^2 - 2$. In this case, we found a much larger number of orbits, so we only report the associated periodic λ-continued fractions up to negation and reversal in Table 3. All of the λ-continued fraction expansions we computed were either finite or

TABLE 1. $q = 7$: Counts of elements of $\mathbb{Z}$, $\mathcal{O}_{K_q}^*$, and $\mathcal{O}_{K_q}$ by G_q-orbit

Orbit representative	$\mathbb{Z} \cap [1, 10^6]$	$\lambda^a(-\lambda')^b$ $a, b \in \mathbb{Z} \cap [-10^2, 10^2]$	$a + b\lambda^2$ $a, b \in \mathbb{Z} \cap [-10^3, 10^3]$
∞	999973	28857	3003907
$\lambda^2 - 1$	0	11446	991616
$\frac{37}{7}\lambda^2 + \frac{29}{7}\lambda - \frac{13}{7}$	0	39	3542
$-\frac{37}{7}\lambda^2 - \frac{29}{7}\lambda + \frac{13}{7}$	0	39	3542
$\frac{43}{7}\lambda^2 + \frac{32}{7}\lambda - \frac{31}{7}$	0	10	645
$-\frac{43}{7}\lambda^2 - \frac{32}{7}\lambda + \frac{31}{7}$	0	10	645
$26\lambda^2 + \frac{39}{2}\lambda - \frac{31}{2}$	15	0	38
$\frac{782}{7}\lambda^2 + \frac{636}{7}\lambda - \frac{428}{7}$	8	0	28
$-\frac{782}{7}\lambda^2 - \frac{636}{7}\lambda + \frac{428}{7}$	4	0	28
$\frac{825}{43}\lambda^2 + \frac{689}{43}\lambda - \frac{375}{43}$	0	0	3
$-\frac{825}{43}\lambda^2 - \frac{689}{43}\lambda + \frac{375}{43}$	0	0	3
$\frac{529}{7}\lambda^2 + \frac{401}{7}\lambda - \frac{313}{7}$	0	0	2
$-\frac{529}{7}\lambda^2 - \frac{401}{7}\lambda + \frac{313}{7}$	0	0	2

TABLE 2. $q = 9$: Counts of elements of $\mathbb{Z}$, $\mathcal{O}_{K_q}^*$, and $\mathcal{O}_{K_q}$ by G_q-orbit

Orbit representative	$\mathbb{Z} \cap [1, 10^6]$	$\lambda^a(-\lambda')^b$ $a, b \in \mathbb{Z} \cap [-10^2, 10^2]$	$a + b\lambda^2$ $a, b \in \mathbb{Z} \cap [-10^3, 10^3]$
Cusps	676292	27225	2707373
$2\lambda + 2$	152442	6247	611123
$-2\lambda - 2$	152957	6247	611123
$8\lambda + 8$	8816	324	35528
$-8\lambda - 8$	8650	324	35528
$\frac{5}{2}\lambda^2 2 + \frac{9}{2}\lambda + \frac{3}{2}$	399	16	1604
$-\frac{5}{2}\lambda^2 2 - \frac{9}{2}\lambda - \frac{3}{2}$	412	16	1604
$\frac{80}{57}\lambda^2 - \frac{74}{57}\lambda - \frac{86}{19}$	19	1	59
$-\frac{80}{57}\lambda^2 + \frac{74}{57}\lambda + \frac{86}{19}$	13	1	59

preperiodic. Moreover, all of the periodic λ-continued fractions $\overline{[a_1, a_2, \ldots, a_n]}$ we found satisfied the following properties:

- n is divisible by 4
- for $1 \leq j \leq n$, there is $k \geq 0$ such that $a_j = \pm 2^k$
- the number of positive and negative a_j's is equal
- $\prod_{j=1}^n a_j = 2^n$

Note that only finitely many periodic parts of each length satisfy the above properties. We systematically checked all periodic parts of length 4, 8, and 12 satisfying the above properties, and listed the periodic parts up to negation and reversal that yield special hyperbolic fixed points in Table 3.

G_q-orbits for other q. For other values of q, we did some ad hoc searches to find additional special hyperbolic fixed points, which are reported in Table 1 in terms of periodic λ-continued fractions. For space reasons, we sometimes use e.g. $(-1)^9$ to denote 9 consecutive occurrences of -1. Note that several entries in Table 1 previously appeared in [AS] and [HMTY].

For $q = 7, 9, 18$, our computations provide strong evidence that every element of $\lambda\mathbb{Q}(\lambda^2)$ is fixed by a nontrivial element of G_q. This was previously conjectured in the cases $q = 7, 9$ in [Bou] and [HMTY]. For other values of q, it seems more difficult to investigate this question empirically.

For simplicity, suppose that K_q has class number 1. Then every element of $\mathbb{P}^1(K_q)$ can be expressed as $[a : b]$ with $a, b \in \mathcal{O}_{K_q}$ relatively prime. Define a *height* on $\mathbb{P}^1(K_q)$ by

$$h([a : b]) = \prod_{\sigma : K_q \hookrightarrow \mathbb{R}} (|\sigma(a)| + |\sigma(b)|) \tag{13}$$

where the product is over the real embeddings of K_q. Note that K_q is totally real. The function h is well-defined since any other expression for $[a : b] \in \mathbb{P}^1(K_q)$ differs by simultaneous multiplication of a, b by a unit $u \in \mathcal{O}_{K_q}^*$, which satisfies $\prod_\sigma |\sigma(u)| = 1$. For elements $x \in \lambda\mathbb{Q}(\lambda^2)$, we will consider the height of x/λ.

Figure 1 illustrates the behavior of $\log(h)$ of a "typical" element of $\lambda\mathbb{Q}(\lambda^2)$ under the λ-continued fraction algorithm. The cases $q = 7, 9, 18$ all behave as in the left image. The cases $q = 14, 30$ behave as in the middle image. The remaining cases (where K_q has degree at least 3) behave as in the right image.

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TABLE 3. $q = 18$, periodic λ-continued fractions in $\lambda\mathbb{Q}(\lambda^2)$

$\overline{[2, 2, -2, -2]}$
$\overline{[4, 1, -4, -1]}$
$\overline{[4, 2, -1, -2]}$
$\overline{[8, 1, -2, -1]}$
$\overline{[16, 1, -1, -1]}$
$\overline{[4, 2, -2, -1, 4, 1, -2, -2]}$
$\overline{[4, 2, -4, -1, 2, 2, -2, -1]}$
$\overline{[8, 1, -4, -1, 8, -1, -1, 1]}$
$\overline{[8, 2, -8, -1, 1, 2, -1, -1]}$
$\overline{[16, 1, -2, -2, 1, 2, -2, -1]}$
$\overline{[4, 2, 2, -1, -2, 2, -4, -2, 2, 1, -2, -2]}$
$\overline{[4, 2, -2, -2, 2, 1, -4, -1, 2, 2, -2, -2]}$
$\overline{[4, 2, -2, -1, 2, 2, -4, -2, -2, 1, 2, -2]}$
$\overline{[4, 2, -4, -1, 2, 1, -4, -2, 4, -1, -2, 1]}$
$\overline{[4, 2, -4, 1, 2, -1, -4, -2, 4, 1, -2, -1]}$
$\overline{[4, 2, -4, -1, 4, 1, -2, -2, 2, 1, -4, -1]}$
$\overline{[4, 4, -4, 1, 2, -1, -4, -1, 4, 1, -2, -1]}$
$\overline{[4, 4, -4, -1, 2, 1, -4, -1, 4, -1, -2, 1]}$
$\overline{[4, 4, -4, -2, -1, 2, 1, -4, -1, 2, 1, -2]}$
$\overline{[8, 2, -2, -1, 2, 2, -2, -2, -2, 1, 2, -2]}$
$\overline{[8, 2, 2, -1, -2, 2, -2, -2, 2, 1, -2, -2]}$
$\overline{[8, 2, 1, -2, -1, 2, -8, -2, -1, 2, 1, -2]}$
$\overline{[8, 2, -8, -1, -2, 1, 2, -2, -2, 1, 2, -1]}$
$\overline{[8, 2, -4, 1, 1, -2, -2, 2, 1, -1, -4, -2]}$
$\overline{[8, 1, -4, -1, 8, 1, -2, -2, 1, 2, -2, -1]}$
$\overline{[16, 2, -2, -1, 2, 2, -1, -2, -2, 1, 2, -2]}$
$\overline{[16, 2, 2, -1, -2, 2, -1, -2, 2, 1, -2, -2]}$
$\overline{[16, 1, 2, -1, -2, 1, -16, -1, -2, 1, 2, -1]}$
$\overline{[16, 1, -2, -1, 8, -1, -1, 1, 8, 1, -2, -1]}$
$\overline{[16, 1, -4, -2, 1, 2, -2, -1, 8, -1, -1, 1]}$
$\overline{[16, -1, -4, 2, 1, -2, -2, 1, 8, 1, -1, -1]}$
$\overline{[32, 1, 1, -1, -4, 1, -8, -1, -4, 1, 1, -1]}$

TABLE 4. Periodic λ-continued fractions in $\lambda\mathbb{Q}(\lambda^2)$

$q=7$	
λ^2-1	$\overline{[1,-1]}$
$\frac{37}{7}\lambda^2+\frac{29}{7}\lambda-\frac{13}{7}$	$\overline{[13,1,2,-13,-2,-1]}$
$\frac{529}{7}\lambda^2+\frac{401}{7}\lambda-\frac{313}{7}$	$\overline{[169,1,2,-1,-2,-1]}$
$\frac{531}{7}\lambda^2+\frac{402}{7}\lambda-\frac{319}{7}$	$\overline{[169,2,1,-1,-1,-2]}$
$\frac{825}{43}\lambda^2+\frac{689}{43}\lambda-\frac{375}{43}$	$\overline{[46,1,-1,2,1,4,-2,-1]}$
$26\lambda^2+\frac{39}{2}\lambda-\frac{31}{2}$	$\overline{[58,1,-1,1,2,2,1,-1,1,-58,-1,1,-1,-2,-2,-1,1,-1]}$
$\frac{782}{7}\lambda^2+\frac{636}{7}\lambda-\frac{428}{7}$	$\overline{[258,-1,1,2,-1,-1,1,-1,1,3,1,1,-1}$ $\overline{-1,-6,7,-2,-1,-1,-2,-1,-7,1]}$
$q=9$	
$2\lambda+2$	$\overline{[3,-4,1,1]}$
$8\lambda+8$	$\overline{[12,-1,3,1,-2,-18,-1,40,3,1^3]}$
$\frac{5}{2}\lambda^2+\frac{9}{2}\lambda+\frac{3}{2}$	$\overline{[10,83,-2,(-1)^3,2,1,4,-1,-1,-4,1,-1]}$
$q=14$	
$\lambda^3-3\lambda$	$\overline{[1,1,-1,-1]}$
$2\lambda^5-6t\lambda^3+3\lambda$	$\overline{[9,-3,-1,-2,-1,-2,-1,-2,-9,3,1,2,1,2,1,2]}$
$10\lambda^5-32\lambda^3+18\lambda$	$\overline{[41,2,-1,-3,(-1)^4 2,1^4,-1]}$
$q=16$	
$\lambda^3-3\lambda$	$\overline{[1,2,1,2,-1,-2,-1,-2]}$
$q=20$	
$\lambda^5-4\lambda^3+2\lambda$	$\overline{[2,1^3,-2,(-1)^3]}$
$2\lambda^3-6\lambda$	$\overline{[2,1,-1,-2,(-1)^9,4,13,1,1,-1,-1,-2,-1,3,-1,}$ $\overline{5,1,1,2,1,-6,(-1)^2,4,(-1)^6,1,2,1^5,-2,1^8]}$
$q=24$	
$\lambda^5-3\lambda^3+\lambda$	$\overline{[5,1,1,-6,(-1)^4,1,1,-5,-1,-1,6,1^4,-1,-1]}$
$\frac{1}{2}\lambda^7-3\lambda^5+\frac{11}{2}\lambda^3-\frac{5}{2}\lambda$	$\overline{[3,-2,-1,1^{11},-3,(-1)^{10},1]}$
$q=30$	
$\lambda^7-6\lambda^5+10\lambda^3-4\lambda$	$\overline{[4,1^5,-4,(-1)^5]}$
$\frac{52}{29}\lambda^7-\frac{207}{29}\lambda^5+\frac{243}{29}\lambda^3-\frac{90}{29}\lambda$	$\overline{[29,-1,-1,-2,(-1)^3,1,-1,-2,-1,3,1^3,4,2,-1,1^{10},}$ $\overline{18,1,-5,2,-2,(-1)^3,1^4-1,1,3,1^5,4,1,1,-19,(-1)^8,}$ $\overline{-2,1^5,5,-29,1,1,2,1^3,-1,1,2,1,-3,(-1)^3,-4,-2,1,}$ $\overline{(-1)^{10},-18,-1,5,-2,2,1^3,(-1)^4,1,-1,-3,(-1)^5,}$ $\overline{-4,-1,-1,19,1^8,2,(-1)^5,-5]}$

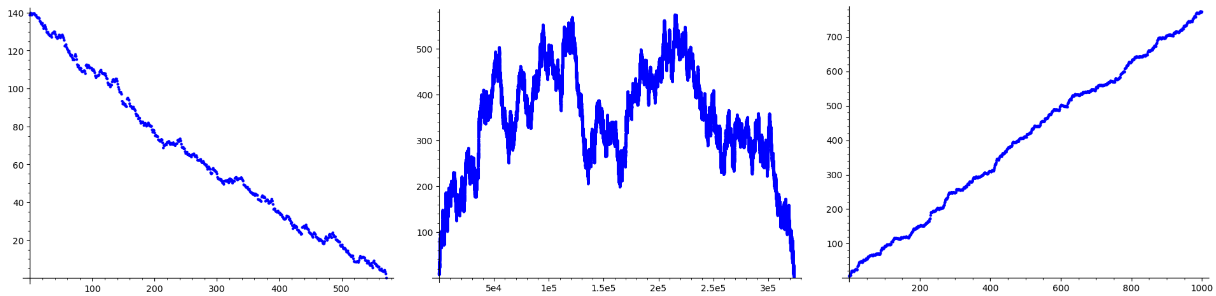

Figure 1. Left: $q = 7$, a plot of $\log h(x)$ for a randomly chosen $x \in \mathbb{Z} + \mathbb{Z}\lambda_7^2$ with $x \approx 10^{30}$. Middle: $q = 14$, a plot of $\log h(x)$ for $x = 32\lambda_{14}^3$. Right: $q = 11$, a plot of $\log h(x)$ for $x = 2$.

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